STRUCTURES AND TRANSFORMATIONS

Automorphism of a structure preserves the structure

Conversely, if we start with the automorphism group, we can extract the invariants, which we can identify with the structure.

cp. Klein's Erlangen Programme in Geometry

Geometry is the study of the invariants of some specified transformation group.
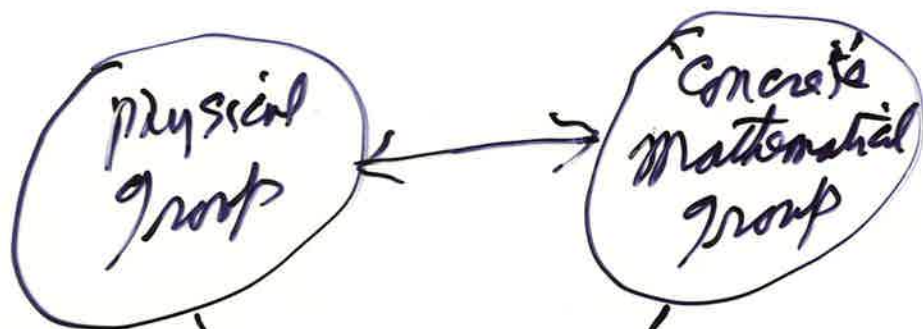
e.g. Euclidean geometry ⟷ Euclidean group
Affine geometry ⟶ Affine "
Projective geometry ⟷ Projective "

More generally, morphisms between structures is the subject matter of Category theory.

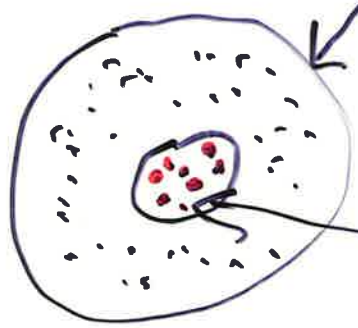# Semantic Approach
to Theories in Mathematical Physics

Ex

Both satisfy the __formal__ axioms of an abstract group presented syntactically

A THEORY OF THEORIES

Possible states of a physical
system associated with points
in a function space.
Physical laws impose constraints
on possible states (functions)

Ex   Solutions of a mathematical
     equation

All possible
functions

Solution space
of allowed
functions.

Symmetries of the physical theory
are automorphisms of the solution
space, i.e. map one solution onto
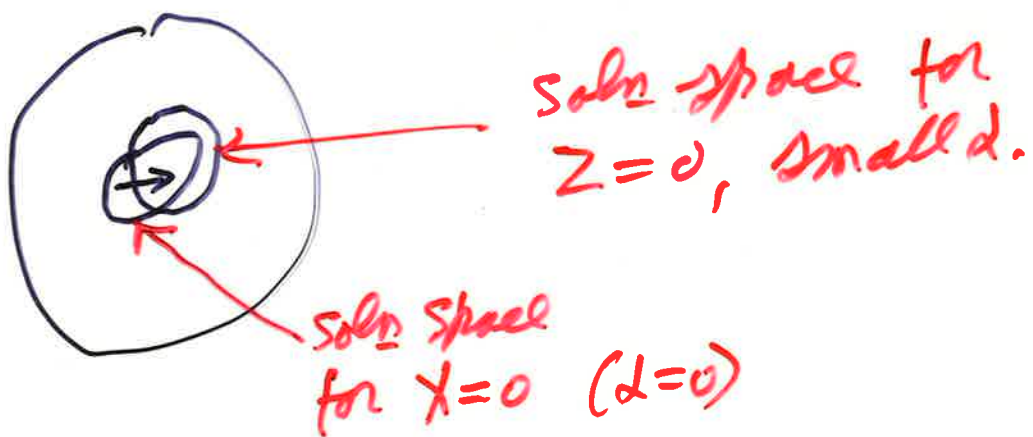another.

Ex $\quad Z = X + \alpha Y = 0 \quad$ (1)

For $\alpha = 0$, eq. becomes.

$$X = 0 \quad (2)$$

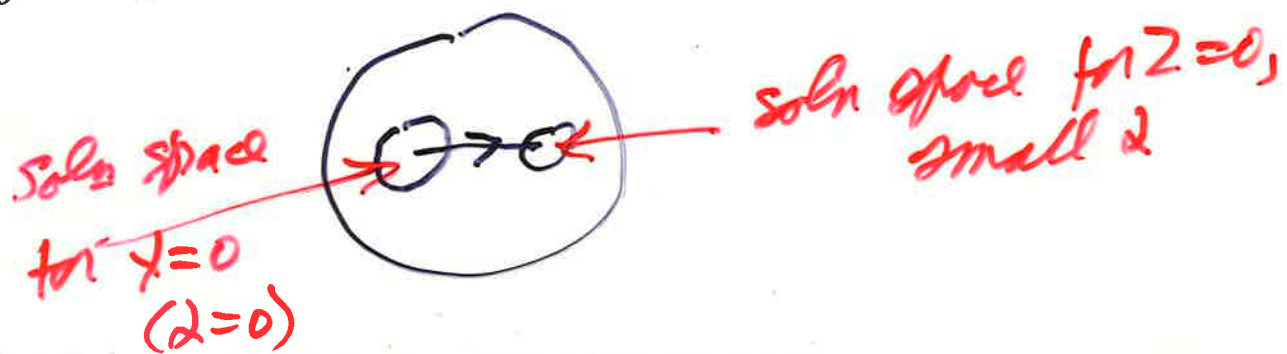How are solutions of (1) for _small_
α related to solutions of (2) where
$\underline{\alpha = 0}$.

Two possibilities:

(a) Continuous behaviour

soln space for
$Z = 0$, small α.

soln space
for $X = 0$ $(\alpha = 0)$

(b) Discontinuous behaviour

soln space for $Z = 0$,
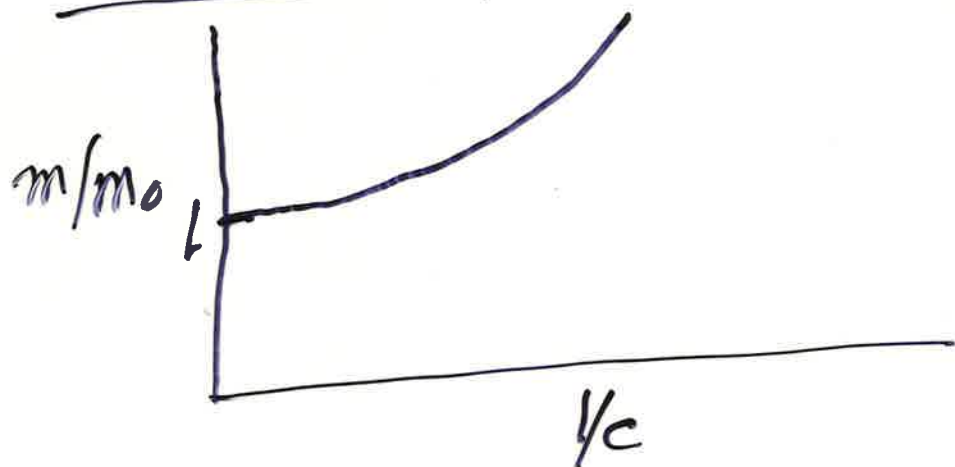small α

soln space
for $X = 0$
$(\alpha = 0)$

Special Relativity as $1/c \to 0$

(a) Consider $m/m_0 = 1/\sqrt{1 - v^2/c^2}$

For fixed v, this behaves in a __continuous__ fashion as $1/c \to 0$

(2) Consider covariant metric tensor
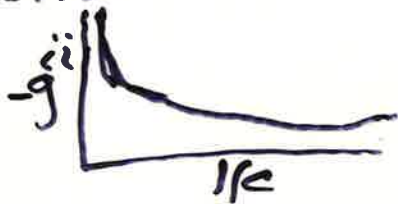
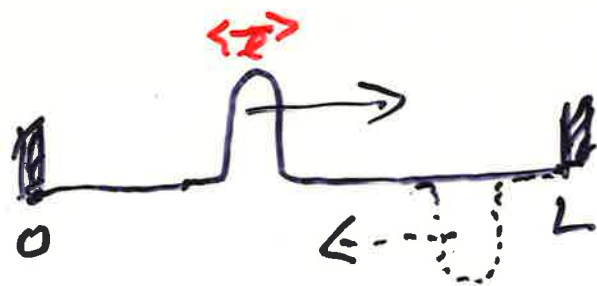$$g_{\mu\nu} = \begin{pmatrix} 1 & -1/c^2 & & \\ & & -1/c^2 & \\ & & & -1/c^2 \end{pmatrix}$$

with $1/c = 0$ this becomes

$$\begin{pmatrix} 1 & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

which has no inverse, so contravariant metric tensor $g^{\mu\nu}$ does not exist! g^{ii} exhibits discontinuity at $1/c = 0$

Consider pulse of length ℓ, unit amplitude travelling to the right

For resolution time T, mean displacement $\bar{\phi}$ satisfies

$$\bar{\phi} < \frac{\ell}{cT} \quad , \quad \text{for } T > \frac{\ell}{c}$$

$$< 1$$

Indeed for **fixed** T, as $\frac{\ell}{c} \to 0$

$$\bar{\phi} \to 0$$

But for any value of $\frac{\ell}{c}$, however small, we can always choose T small enough $(< \ell/c)$ so as to make $\bar{\phi} = 1$ for some value of x and t.

Compare fundamental mode with the zero **soln** at a particular point.

$$\int_0^T (\sin \nu t - 0)\, dt = \frac{1}{\nu}(1 - \cos \nu T) \to 0$$
$$\text{as } \nu = \frac{c}{2L} \to \infty.$$

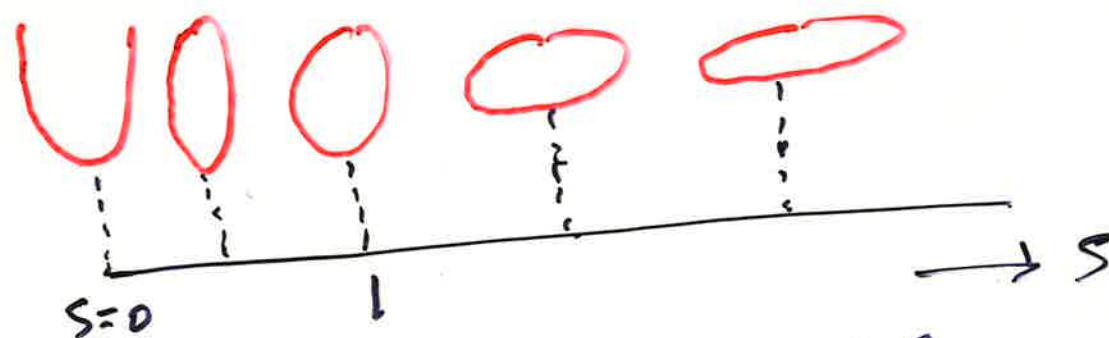So we have converged **in the mean**, but not **pointwise convergence**.

Conic Sections and Structural Stability

Consider the following eq. in 2-dimensional Euclidean space

$$x^2 + s y^2 - 2x - 2y + 1 = 0$$

for variable parameter $s \geqslant 0$

For $s = 0$ we have a parabola
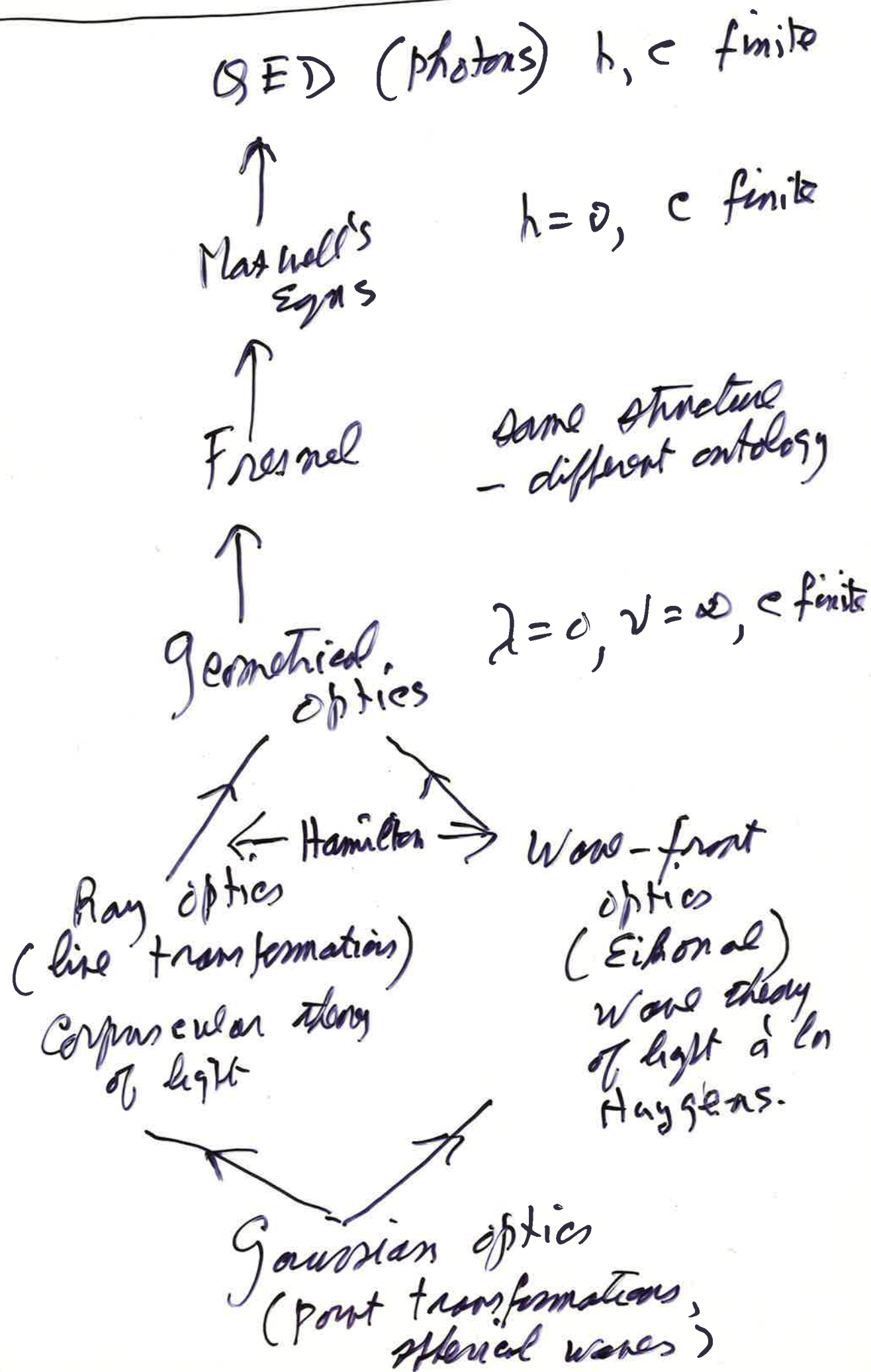For $s > 0$ ellipses of varying eccentricity ($s = 1$ gives a circle).

What we are dealing with here is a structure-valued function over Relative to the Euclidean group all the figures are in different congruence classes.

But relative to the Affine group all the ellipses are equivalent, but not the parabola, so we have structural stability relative to the affine group for all $s > 0$, with a singularity at $s = 0$ (catastrophe)
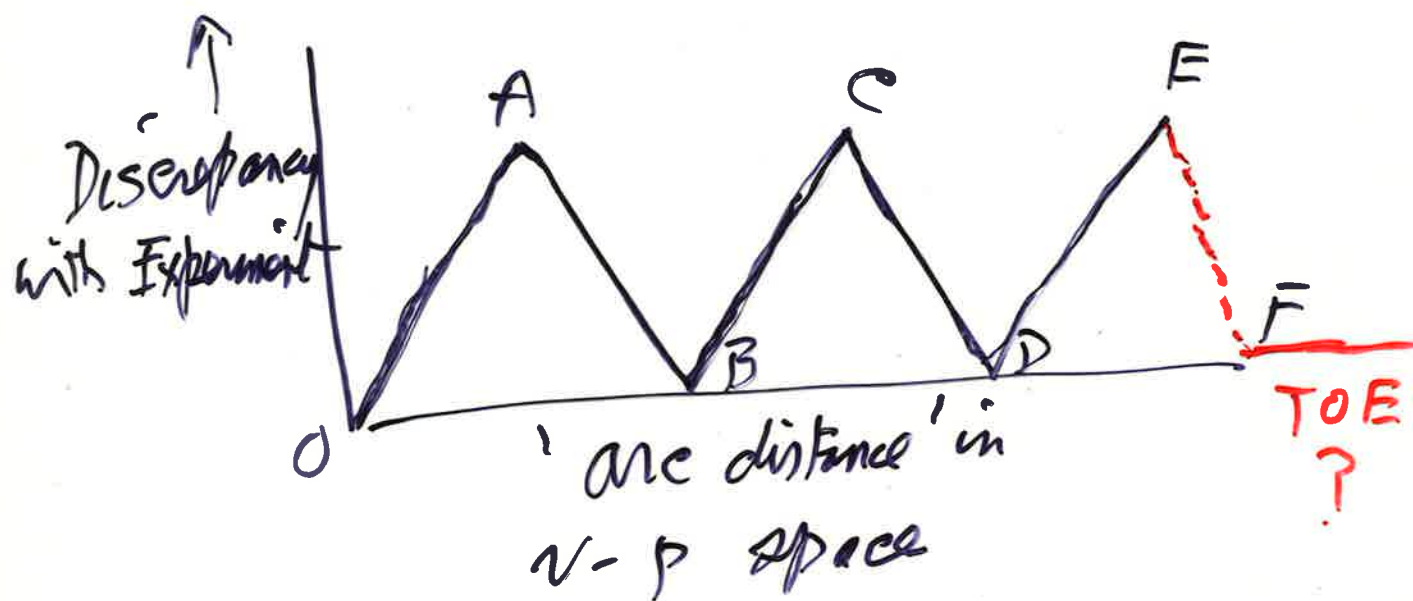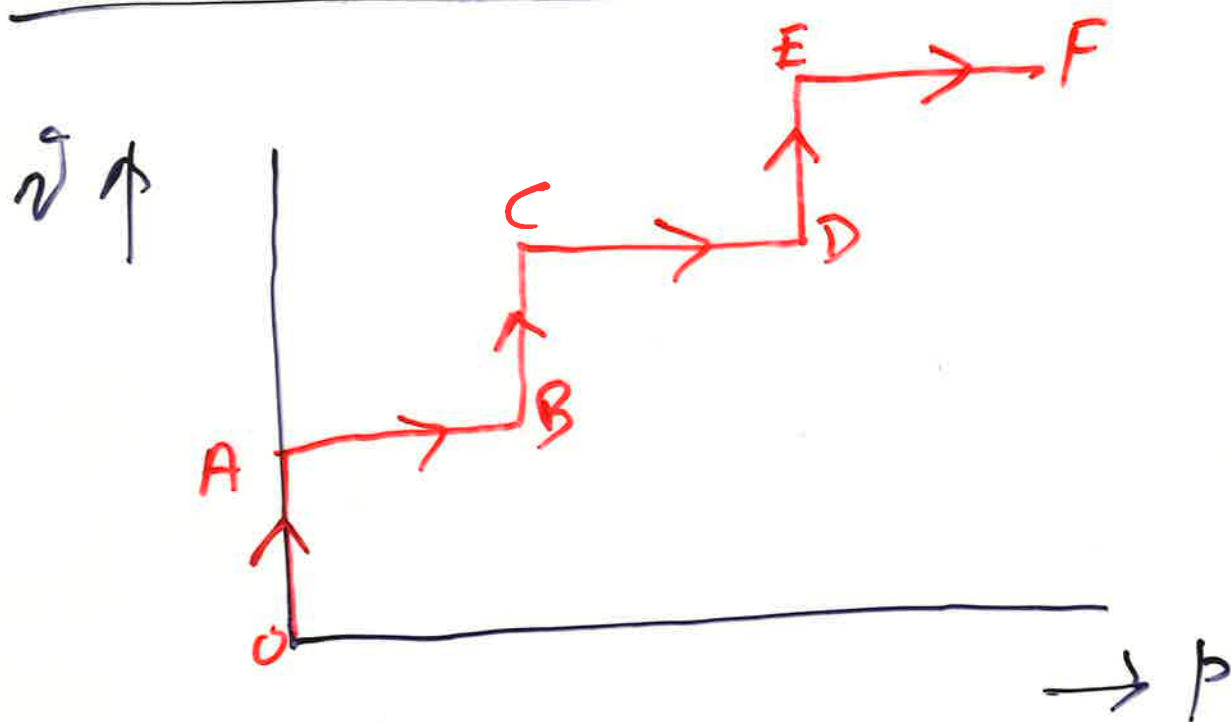
QED (photons) h, c finite

$\uparrow$

Maxwell's Eqns $h = 0, c$ finite

$\uparrow$

Fresnel   same structure — different ontology

$\uparrow$

Geometrical optics $\lambda = 0, \nu = \infty, c$ finite

$\swarrow \qquad \searrow$

$\longleftarrow$ Hamilton $\longrightarrow$

Ray optics   Wave-front optics
(line transformations)   (Eikonal)

Corpuscular theory of light   Wave theory of light à la Huygens.

$\searrow \qquad \nwarrow$

Gaussian optics
(point transformations, spherical waves)

PROGRESS IN PHYSICS

'arc distance' in N-p space

Gauge Transformations and Surplus Structure

$\rho = e \psi^* \psi$ and $j = \frac{1}{2} i e \left(\psi^* \frac{d\psi}{dx} - \psi \frac{d\psi^*}{dx} \right)$

are real quantities and represent physical magnitudes of charge and current density.

ρ and j are invariant under global phase transformations $\psi \to \psi e^{i\alpha}$

But to retain local gauge invariance under $\psi \to \psi e^{i\alpha(x)}$ for the current density j, we must replace d/dx by $d/dx - i A(x)$ where A transforms as $A \to A + d/dx \, \alpha(x)$ and $j = \frac{1}{2} \cdot i e \left(\psi^* (d/dx - iA) \psi - \psi (\frac{d}{dx} + iA) \psi^* \right)$